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SOME RECURRENCE FORMULAS FOR A NEW CLASS OF SPECIAL POLYNOMIALS AND SPECIAL FUNCTIONS

Abstract:

In this paper we used a new class of special functions and special polynomials which are solutions different Sturm Liouvile differential equations of second order. These functions form a basis of a space of square integrable functions over set of a real numbers. We investigated some properties of these polynomials and established some recurrence formulas. Using a new class of special functions, we obtained some useful summation formulas and recurrence formulas.

Keywords: differential equations, recurrence formulas, special functions, special polynomials

НЕКЕ РЕКУРЗИВНЕ ФОРМУЛЕ ЗА НОВУ КЛАСУ СПЕЦИЈАЛНИХ ПОЛИНОМА И СПЕЦИЈАЛНИХ ФУНКЦИЈА

Сажетак:

У раду смо користили нову класу специјалних функција и специјалних полинома који су рјешења различитих Штурм Лиувилових диференцијалних једначина другог реда. Те функције формирају базу простора квадратно интеграбилних функција. Испитали смо неке особине таквих полинома и добили рекурзивне релације са њима. Користећи нову класу специјалних функција добили смо корисне формуле за сумирање, као и рекурзивне релације са таквим функцијама.

Кључне ријечи: диференцијалне једначине, рекурзивне формуле, специјалне фунције, специјални полиноми

1. INTRODUCTION

Orthogonal polynomials (Hermite, Laguerre, etc, [1-6]) and orthonormal functions (Hermite, Laguerre, etc [7], [8]) are very useful and have many applications in engineering. Orthogonal polynomials play an important role in the study of wave mechanics, heat conduction, electromagnetic theory, quantum mechanics and mathematical statistics. Some of those polynomials are used in the field of construction, for example for the beam bending test [9]. Hermite polynomials are readily applied in computational mechanics for a spatial discretization of the kinematic field. Standard applications include structural analysis of beams [10] and shells [11].

A recurrence relation is an equation that defines a sequence based on a rule that gives the next term as a function of the previous term (or terms). The simplest form of a recurrence relation is the case where the next term depends only on the immediately previous term. A recurrence relation can also be higher order *k*>2, where the term a_{n+1} could depend not only on the previous term a_n but also on earlier terms $a_{n-1}, a_{n-2}, \dots, a_{n-k+1}$ [12]. *Recursive techniques are very useful for calculation.* In [4] are given many recursive relations with orthogonal polynomials (orthogonal polynomials have three order recursive relations). Motivation for this paper is to establish recurrence formulas for a new class of special polynomials and special functions introduced in [2].

A special polynomials of the form

$$F_0(x) = 1$$

$$F_{2n}(x) = Re((x-i)^{2n}) = \sum_{k=0}^{n}(-1)^{n+k}\binom{2n}{2k}x^{2k} \tag{1}$$

$$F_{2n-1}(x) = Im((x-i)^{2n}) = \sum_{k=1}^{n}(-1)^{n+k+1}\binom{2n}{2k-1}x^{2k-1}$$

are studied in [3]. Although they are non-orthogonal, they play important role because are enumerators of rational functions of the form

$$f_{2n}(x) = \frac{F_{2n}(x)}{(x^2+1)^{n+1/2}},\ f_{2n-1}(x) = \frac{F_{2n-1}(x)}{(x^2+1)^{n+1/2}},\ n \in N\ . \tag{2}$$

It is shown in [2] that these functions are solutions of the Sturm-Liouville differential equation

$$(1+x^2)^2y''(x) + 4x(1+x^2)y'(x) + (1+2x^2+4n^2)y(x) = 0$$

and form an orthonormal basis of $L^2(R)$ space.

2. PRELIMINARIES AND NOTATION

We use the following notation: N, R, C for sets of natural, real and complex numbers, respectively. By *i* we denote the imaginary unit, *Re(z)* and *Im(z)* means the real part and the imaginary part of the complex number z, respectively. By $L^2(R)$ [13] we mean the space of square integrable functions over R, that is

$$L^2(R) = \left\{f: R \to C \,\middle|\, \int_{-\infty}^{\infty}|f(x)|^2dx < \infty\right\}$$

In [3] are determined generating functions for polynomials $F_n(x), n \in N$ and obtained very useful and interesting summation formulas. Also, it is proven in [3] that special polynomials $F_n(x), n \in N$ satisfy ($\forall x, t \in$ R)

$$\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{(2n)!}t^{2n} = \cos t\cosh(xt) \tag{3}$$

$$\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)}{(2n)!}t^{2n} = -\sin t\sinh(xt) \tag{4}$$

If we put $t = 1$ in (3) and (4) we obtain

$$e^x = \sinh(x) + \cosh(x)$$

$$e^x = \frac{1}{\cos 1}\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{(2n)!} - \frac{1}{\sin 1}\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)}{(2n)!} \tag{5}$$

Below are some formulas proven in [3].

For every *x,t∈R* it holds that

$$\sum_{n=0}^{\infty}(-1)^n\frac{F_{2n}(x)}{(2n)!}t^{2n} = \cos(xt)\cosh(t) \tag{6}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{F_{2n-1}(x)}{(2n)!}t^{2n} = \sin(xt)\sinh(t) \tag{7}$$

If we put $t = 1$ in (6) and (7) we obtain

$$\cos(x) = \frac{1}{\cosh 1}\sum_{n=0}^{\infty}(-1)^n \frac{F_{2n}(x)}{(2n)!} \quad (8)$$

$$\sin(x) = \frac{1}{\sinh 1}\sum_{n=1}^{\infty}(-1)^n \frac{F_{2n-1}(x)}{(2n)!} \quad (9)$$

For every $x,\ t\in R$ it holds that

$$\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n = e^{x^2t-t}\cos(2xt) \quad (10)$$

$$\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)}{n!}t^n = -e^{x^2t-t}\sin(2xt) \quad (11)$$

3. RECURRENCE FORMULAS FOR SPECIAL POLYNOMIALS

Research with a new class of special polynomials is in its infancy. There are no recurrence relations with them, so in this section we give some interesting recursive relations with special polynomials $\boldsymbol{F_n(x), n \in N}$.

Proposition 3.1: For special polynomials $F_n(x), n \in N$ it holds that

$$F_{2n+2}'(x) = 2(n+1)(xF_{2n}(x) + F_{2n-1}(x)) \quad (12)$$

$$F_{2n+2}(x) = (x^2-1)F_{2n}(x) + 2xF_{2n-1}(x) \quad (13)$$

$$xF_{2n}'(x) = n((x^2+1)F_{2n-2}(x) + F_{2n}(x)) \quad (14)$$

Proof: If we derivate equation (10) by x we obtain (we can derivate since it converge for $(\forall x \in \mathrm{R})$)

$$\sum_{n=0}^{\infty}\frac{F_{2n}'(x)}{n!}t^n = (e^{x^2t-t}\cos(2xt))_x' = 2xte^{x^2t-t}\cos(2xt) - 2te^{x^2t-t}\sin(2xt)$$

$$= 2xt\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n + 2t\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)}{n!}t^n = \sum_{n=0}^{\infty}\frac{2xF_{2n}(x)}{n!}t^{n+1} + \sum_{n=1}^{\infty}\frac{2F_{2n-1}(x)}{n!}t^{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{2xF_{2(n-1)}(x)}{(n-1)!}t^n + \sum_{n=2}^{\infty}\frac{2F_{2(n-1)-1}(x)}{(n-1)!}t^n = 2xF_0(x)t + \sum_{n=2}^{\infty}\frac{2xF_{2n-2}(x) + 2F_{2n-3}(x)}{(n-1)!}t^n$$

From above equations we have

$$F_0'(x) + F_2'(x)t + \sum_{n=2}^{\infty}\frac{F_{2n}'(x)}{n!}t^n = 2xF_0(x)t + \sum_{n=2}^{\infty}\frac{2xF_{2n-2} + 2F_{2n-3}}{(n-1)!}t^n$$

Since $\mathrm{F}_0(x) = 1 \Rightarrow F_0'(x) = 0$ and $F_2(x) = x^2 - 1 \Rightarrow F_2'(x) = 2x$ we have

$$\sum_{n=2}^{\infty}\frac{F_{2n}'(x)}{n!}t^n = \sum_{n=2}^{\infty}\frac{2xF_{2n-2}(x) + 2F_{2n-3}(x)}{(n-1)!}t^n.$$

This implies $F_{2n}'(x) = 2n(xF_{2n-2}(x) + F_{2n-3}(x)), n\geq 2$, so we obtain (12). To get (13) we derivate (10) by t, so we have

$$\sum_{n=1}^{\infty}\frac{F_{2n}(x)}{(n-1)!}t^{n-1} = (x^2-1)e^{x^2t-t}\cos(2xt) - 2xe^{x^2t-t}\sin(2xt)$$

$$= (x^2-1)\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n + 2x\sum_{n=1}^{\infty}\frac{F_{2n-1}(x)}{n!}t^n$$

$$\sum_{n=0}^{\infty}\frac{F_{2n+2}(x)}{n!}t^n = (x^2-1)F_0(x) + \sum_{n=1}^{\infty}\frac{(x^2-1)F_{2n}(x) + 2xF_{2n-1}(x)}{n!}t^n$$

$$F_2(x) + \sum_{n=1}^{\infty}\frac{F_{2n+2}(x)}{n!}t^n = (x^2-1)F_0(x) + \sum_{n=1}^{\infty}\frac{(x^2-1)F_{2n}(x) + 2xF_{2n-1}(x)}{n!}t^n$$

Since $F_0(x) = 1$ and $F_2(x) = x^2 - 1$, from the last equality we obtain (13).

Notice that

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n = 2xte^{x^2t-t}\cos 2xt - 2te^{x^2t-t}\sin\ 2xt \quad (15)$$

$$\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n = (x^2-1)e^{x^2t-t}\cos 2xt - 2xe^{x^2t-t}\sin 2xt \quad (16)$$

If we multiply (15) by x and (16) by (-t) we obtain

$$t(x^2+1)e^{x^2t-t}\cos 2xt = x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n - t\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n$$
$$t(x^2+1)\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^n = x\sum_{n=0}^{\infty}\frac{F'_{2n}(x)}{n!}t^n - t\sum_{n=1}^{\infty}\frac{F_{2n}(x)}{(n-1)!}t^{n-1}$$
$$(x^2+1)\sum_{n=0}^{\infty}\frac{F_{2n}(x)}{n!}t^{n+1} = \sum_{n=0}^{\infty}\frac{xF'_{2n}(x)}{n!}t^n - \sum_{n=1}^{\infty}\frac{F_{2n}(x)}{(n-1)!}t^n$$

$$(x^2+1)\sum_{n=1}^{\infty}\frac{F_{2n-2}(x)}{(n-1)!}t^n = xF_0'(x) + \sum_{n=1}^{\infty}\frac{\frac{xF'_{2n}(x)}{n} - F_{2n}(x)}{(n-1)!}t^n$$

Since $F_0'(x) = 0$ we obtain (14).

Proposition 3.2: For special polynomials $F_n(x), n \in N$ it holds that

$$F'_{2n+1}(x) = 2(n+1)(xF_{2n-1}(x) - F_{2n}(x)) \quad (17)$$
$$F_{2n+1}(x) = (x^2-1)F_{2n-1}(x) - 2xF_{2n}(x) \quad (18)$$
$$xF'_{2n+1}(x) = (n+1)((x^2+1)F_{2n-1}(x) + F_{2n+1}(x)) \quad (19)$$

Proof: The proof is similar like those in Proposition 3.1. To obtain (17) we derivate (11) by x, and to get (18) we derivate (11) by t. Relation (19) we obtain on the same way as in Proposition 3.1 (see (15) and (16)).

4. RECURRENCE FORMULAS FOR SPECIAL FUNCTIONS

In this section we give some generating functions for a new class of special functions $f_n(x), n \in N_0$. Using these sumation formulas we obtain interesting recursive formulas for $f_n(x), n \in N_0$.

Proposition 4.1: For special functions $f_n(x), n \in N_0$ and $|t| < 1$ it holds that

$$\sum_{n=0}^{\infty} f_{2n}(x)t^n = \frac{1}{\sqrt{x^2+1}}\frac{x^2(1-t)+1+t}{x^2(1-t)^2+(1+t)^2} \quad (20)$$

$$\sum_{n=1}^{\infty} f_{2n-1}(x)t^n = -\frac{1}{\sqrt{x^2+1}}\frac{2xt}{x^2(1-t)^2+(1+t)^2} \quad (21)$$

Proof: We use geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$. Taking real and imaginary parts in

$$\sum_{n=0}^{\infty} f_n(x)t^n = \frac{1}{\sqrt{x^2+1}}\sum_{n=0}^{\infty}\left(\frac{x-i}{x+i}\right)^n t^n = \frac{1}{\sqrt{x^2+1}}\frac{1}{1-\frac{x-i}{x+i}t}$$
$$= \frac{x+i}{\sqrt{x^2+1}\big((x-xt)+i(1+t)\big)} = \frac{1}{\sqrt{x^2+1}}\left(\frac{x^2(1-t)+1+t}{x^2(1-t)^2+(1+t)^2}\right)$$
$$-i\frac{1}{\sqrt{x^2+1}}\left(\frac{2xt}{x^2(1-t)^2+(1+t)^2}\right)$$

we obtain (20) and (21).

Proposition 4.2: For special functions $f_n(x), n \in N_0$ it holds that $(\forall t \in R)$

$$\sum_{n=0}^{\infty}\frac{f_{2n}(x)}{n!}t^n = \frac{1}{\sqrt{x^2+1}}e^{\frac{x^2-1}{x^2+1}t}\cos\frac{2xt}{x^2+1} \quad (22)$$

$$\sum_{n=1}^{\infty}\frac{f_{2n-1}(x)}{n!}t^n = -\frac{1}{\sqrt{x^2+1}}e^{\frac{x^2-1}{x^2+1}t}\sin\frac{2xt}{x^2+1} \quad (23)$$

Proof: Taking real and imaginary parts in

$$\sum_{n=0}^{\infty}\frac{f_n(x)}{n!}t^n = \frac{1}{\sqrt{x^2+1}}\sum_{n=0}^{\infty}\left(\frac{x-i}{x+i}\right)^n\frac{t^n}{n!} = \frac{1}{\sqrt{x^2+1}}e^{\frac{x-i}{x+i}t} = \frac{1}{\sqrt{x^2+1}}e^{\frac{x^2-1}{x^2+1}t-\frac{2xt}{x^2+1}i}$$
$$= \frac{1}{\sqrt{x^2+1}}e^{\frac{x^2-1}{x^2+1}t}\left(\cos\frac{2xt}{x^2+1} - i\sin\frac{2xt}{x^2+1}\right)$$

we obtain (22) and (23).

Proposition 4.3: For special functions $f_n(x), n \in N_0$ it holds that

$$f_{2n+2}(x) = \frac{x^2-1}{x^2+1} f_{2n}(x) + \frac{2x}{x^2+1} f_{2n-1}(x) \tag{24}$$

$$f_{2n+1}(x) = \frac{x^2-1}{x^2+1} f_{2n-1}(x) - \frac{2x}{x^2+1} f_{2n}(x) \tag{25}$$

Proof: If we derivate (21) by $t \in R$, we obtain

$$\sum_{n=1}^{\infty} \frac{f_{2n}(x)}{(n-1)!} t^{n-1} = \frac{x^2-1}{(x^2+1)^{\frac{3}{2}}} e^{\frac{x^2-1}{x^2+1}t} \cos\frac{2xt}{x^2+1} - \frac{2x}{(x^2+1)^{\frac{3}{2}}} e^{\frac{x^2-1}{x^2+1}t} \sin\frac{2xt}{x^2+1}$$

$$= \frac{x^2-1}{x^2+1} \sum_{n=0}^{\infty} \frac{f_{2n}(x)}{n!} t^n + \frac{2x}{x^2+1} \sum_{n=1}^{\infty} \frac{f_{2n-1}(x)}{n!} t^n$$

$$\sum_{n=0}^{\infty} \frac{f_{2n+2}(x)}{n!} t^n = \frac{x^2-1}{x^2+1} \sum_{n=0}^{\infty} \frac{f_{2n}(x)}{n!} t^n + \frac{2x}{x^2+1} \sum_{n=1}^{\infty} \frac{f_{2n-1}(x)}{n!} t^n$$

$$f_2(x) + \sum_{n=1}^{\infty} \frac{f_{2n+2}(x)}{n!} t^n = \frac{x^2-1}{x^2+1} f_0(x) + \sum_{n=1}^{\infty} \left(\frac{x^2-1}{x^2+1} f_{2n}(x) + \frac{2x}{x^2+1} f_{2n-1}(x) \right) \frac{t^n}{n!}$$

Since $f_0(x) = \frac{1}{\sqrt{x^2+1}}, f_2(x) = \frac{x^2-1}{(x^2-1)^{\frac{3}{2}}}$ we obtain (24).

If we derivate (23) by $t \in R$, we obtain

$$\sum_{n=1}^{\infty} \frac{f_{2n-1}(x)}{(n-1)!} t^{n-1} = -\frac{x^2-1}{(x^2+1)^{\frac{3}{2}}} e^{\frac{x^2-1}{x^2+1}t} \sin\frac{2xt}{x^2+1} - \frac{2x}{(x^2+1)^{\frac{3}{2}}} e^{\frac{x^2-1}{x^2+1}t} \cos\frac{2xt}{x^2+1}$$

$$\sum_{n=0}^{\infty} \frac{f_{2n+1}(x)}{n!} t^n = \frac{x^2-1}{(x^2+1)^{\frac{3}{2}}} \sum_{n=1}^{\infty} \frac{f_{2n-1}(x)}{n!} t^n - \frac{2x}{x^2+1} \sum_{n=0}^{\infty} \frac{f_{2n}(x)}{n!} t^n$$

$$f_1(x) + \sum_{n=1}^{\infty} \frac{f_{2n-1}(x)}{n!} t^n = \sum_{n=1}^{\infty} \left(\frac{x^2-1}{x^2+1} f_{2n-1}(x) - \frac{2x}{x^2+1} f_{2n}(x) \right) \frac{t^n}{n!} - \frac{2x}{x^2+1} f_0(x)$$

Since $f_1(x) = \frac{-2x}{(x^2+1)^{\frac{3}{2}}}$ we obtain (25).

5. CONCLUSION

Recursive relations play important role in calculus. Classical orthogonal polynomials have recurrence formulas of the third order. A new class of non-orthogonal polynomial is not yet investigated and there are not recurrence relations with them. In this paper recurrence formulas (13), (18) of the fourth order with a new class of special polynomials are obtained. Also, we considered a new class of special functions and obtained some generating functions for them (20)-(23) and fourth order recurrence relations (24), (25).

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LITERATURE:

[1] P. Beckmann, Orthogonal Polynomials for Engineers, Golem PrJ, first edition 1973.

[2] N. Đurić, S. Maksimović, “One class of special polynomials and special functions in L2 (R)“, NSJOM, 1, 81-91, 2019.

[3] N. Đurić, S. Maksimović, “Summation formulas using a new class of special polynomials“,2020 19th International Symposium INFOTEH-JAHORINA (INFOTEH), East Sarajevo, Bosnia and Herzegovina, 1-4, 2020.

[4] A. Erdelyi, Higher trancendental functions, McGraw Hill, 1953.

[5] D. Gómez-Ullate, N. Kamran and R. Milson, “An extended class of orthogonal polynomials defined by a Sturm-Liouville problem“. J. Math. Anal. Appl. 359, 1, 352-367, 2009.

[6] G.Szegö, Orthogonal polynomials. American Mathematical Society Colloquium Publications, Vol. 23. Revised ed. American Mathematical Society, Providence, R.I., 1959.

[7] J. Boyd, “Spectral Methods Using Rational Basis Functions on an Infinite interval“, Journal of computational physics , 69, 112-142, 1987.

[8] E.I. Moiseev, A.P. Prudnikov, “On a complete orthonormal system of special functions“, Journal of Computational and Applied Mathematics 49, 201-206, 1993.

[9] F. Pampaloni and J. Enderlein, “Gaussian, Hermite-Gaussian, and Laguerre Gaussian beams: A primer. Oct 2004, physics/0410021 (26.02.2020).

[10] F. Armero, J. Valverde, “Invariant Hermitian finite elements for thin Kirchhoff rods: The linear plane case“, Computer Methods in Applied Maechanics and Engineering, 213-216, 427-457, 2012.

[11] A. Borković, S. Kovačević, D. Milašinović, G. Radenković, O. Mijatović, V. Golubović-Bugarski, “Geometric nonlinear analysis of prismatic shells using the sem-analytical finite strip method“, Thin-Walled Structures, 117, 63-88, 2017.

[12] Nykamp DQ, “Recurrence relation definition.” From Math Insight. http://mathinsight.org/definition/recurrence_relation (26.02.2020).

[13] L. Schwartz, Functional analysis. Courant Institute of Mathematical Sciences, New York University, 1964.